Optimal Surrender Policy for Variable Annuity Guarantees

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Outline

1. Introduction

2. Optimal surrender boundary: GMAB case

3. Optimal exercise boundary: Path-dependent case

4. Conclusion
Variable annuities and financial options

- Variable annuities: similar to mutual funds with additional guarantees

- VA guarantees paid for via a fixed fee rate throughout the life of the contract (not paid upfront)
  - Decreases return on fund
  - Impacts value of option

- VA contract can be surrendered (Knoller, Kraut, and Schoenmaekers (2011))
  - Financial needs
  - Higher costs of opportunity
  - Moneyness of the option
Continuous fee and the surrender option

- Simple payoff at maturity:
  \[
  \max(G, F_T) = F_T + (G - F_T)^+
  \]

- Option paid by continuous fee set as percentage of fund:
  - Fee is low when option value is high
  - *Incentive to surrender* when fund value is high

- Surrender region: surrender benefit higher than payoff expected if contract is kept
Setting

Index value $S_t$:

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t
\]

Account value $F_t$ based on index:

\[
\frac{dF_t}{F_t} = (r - c) \, dt + \sigma \, dW_t
\]

\[\Rightarrow F_t \mid F_s \sim \mathcal{L}N(\log(F_t) + (r - c - \frac{\sigma^2}{2})(t - s), \sigma^2(t - s))\]
Contract

- Accumulation benefit:
  \[ \max(F_T, G), \quad G = F_0 e^{gT}, \quad g < r \]

- Surrender benefit:
  \[ e^{-\kappa(T-t)} F_t \]
Integral representation for surrender option

- Similarities between surrender option and American option
- Can use techniques developed for American options (Kim (1990), Kim and Yu (1996), Carr, Jarrow, and Myneni (1992), Wu and Fu (2003))
- Integral can be obtained in different ways:
  - Finite number of surrender times (as in Kim (1990))
  - No-arbitrage arguments (as in Kim and Yu (1996))
Trading strategy for surrender option

- Confirm that optimal strategy is a **threshold** strategy
- **Hold the VA** below the surrender boundary $B_t$
- When $F_t$ crosses $B_t$ from below, **sell the VA** and invest the proceeds
- When $F_t$ crosses $B_t$ from above, use the investment to **buy the VA**
- Payoff of portfolio is payoff of VA
Gain from surrender

**Proposition**

The benefit associated with the surrender option between $[t, t + dt]$ for an infinitesimal time step $dt$ is given by

$$e^{-\kappa(T-t)}(c - \kappa) F_t dt$$
Proof of Proposition

• Suppose surrender at $t$.

• Policyholder receives

\[
e^{-\kappa(T-t)} F_t = e^{-\kappa(T-t)} e^{-ct} S_t = e^{-\kappa T} e^{-(c-\kappa)t} S_t
\]

• To buy the VA at time $t + dt$, policyholder only needs

\[
e^{-\kappa(T-(t+dt))} F_{t+dt} = e^{-\kappa T} e^{-(c-\kappa)(t+dt)} S_{t+dt}
\]

• Investment made at $t$ becomes

\[
e^{-\kappa T} e^{-(c-\kappa)(t+dt)} S_{t+dt} + e^{-\kappa T} e^{-(c-\kappa)t} S_t e^{rdt} (1 - e^{-(c-\kappa)dt})
\]
\[
= e^{-\kappa(T-(t+dt))} F_{t+dt} + e^{-\kappa(T-t)}(c-\kappa) F_t dt + o(dt)
\]
Price of VA with surrender

Price of VA contract

\[ V(t, F_t) = E[e^{-rt} \max(G, F_T) | \mathcal{F}_t] + \]

Maturity benefit

\[ e^{-\kappa T} (c - \kappa) F_t \int_t^T e^{-(c - \kappa)u} \Phi(d_1(F_t, B_u, u, t)) du \]

Surrender option

- Maturity benefit: Similar to vanilla option under Black-Scholes
- Surrender option:

\[ \int_t^T e^{-r(u-t)} \int_{B_u}^{\infty} e^{-\kappa(T-u)}(c - \kappa) x f_{F_u}(x | F_t) dx du. \]
Optimal exercise boundary condition

- At maturity $B_T = G_T$ and along the surrender boundary, $V(F_t, t) = e^{-\kappa(T-t)}F_t = B_t$.

- Work backwards to solve for $B_t$

\[
B_t = v(F_t, t) + e(F_t, t) = e^{-c(T-t)}B_te^{\kappa(T-t)}\Phi(d_1(B_te^{\kappa(T-t)}, G_T, T, t)) + e^{-r(T-t)}G_T\Phi(d_2(B_te^{\kappa(T-t)}, G_T, T, t)) + (c - \kappa)B_te^{(c-\kappa)t}\int_t^T e^{-(c-\kappa)u}\Phi(d_1(B_te^{\kappa(T-t)}, B_u, u, t))du
\]
Numerical example

Assumptions:

- $T = 15$
- $G = F_0 = 100$
- $\kappa = 0$, unless otherwise indicated
- $c = 0.91\%$ (fair fee for maturity benefit)
- $r = 0.03$
- $\sigma = 0.2$, unless otherwise indicated
Optimal exercise boundary, sensitivity analysis: \( \sigma \)

- \( \sigma = 0.15 \)
- \( \sigma = 0.20 \)
- \( \sigma = 0.25 \)
- \( \sigma = 0.30 \)
Optimal exercise boundary, sensitivity analysis: kappa

- $k=0$
- $k=c/4$
- $k=c/3$
- $k=c/2$
Consider the payoff $\max(G_T, Y_T)$, where $Y_T$ is the geometric average defined as

$$Y_t = \exp\left(\frac{1}{t} \int_0^t \ln F_s \, ds\right)$$

The conditional distribution of $Y_u|(Y_t, F_t)$ for $u > t$ is again log-normal with mean and variance given by

$$M^g_t = \frac{t}{u} \ln Y_t + \frac{u - t}{u} \ln F_t + \frac{r - c - \frac{\sigma^2}{2}}{2u} (u - t)^2$$

$$V^g_t = \frac{\sigma^2}{3u^2} (u - t)^3$$
Pricing formula

**Theorem**

Let $V^g(Y_t, F_t, t)$ denote the price at time $t$ of the VA with guarantee $G_T$ and a surrender benefit equal to $e^{-\kappa(T-t)}Y_t$. Then $V^g(Y_t, F_t, t)$ can be decomposed into a European part $v^g(Y_t, F_t, t)$ and an early exercise premium $e^g(Y_t, F_t, t)$

$$V^g(Y_t, F_t, t) = v^g(Y_t, F_t, t) + e^g(Y_t, F_t, t),$$

where

$$v^g(Y_t, F_t, t) = e^{-r(T-t)}e^{M_t^g + \frac{V_t^g}{2}}\Phi\left(-\ln(G_T)+\frac{M_t^g+V_t^g}{\sqrt{V_t^g}}\right) + e^{-r(T-t)}G_T\Phi\left(\frac{\ln(G_T)-M_t^g}{\sqrt{V_t^g}}\right),$$

$$e^g(Y_t, F_t, t) = e^{-\kappa T}e^{rt} \int_t^T e^{u(\kappa-r)}e^{-\frac{V_{u,t}}{2}}Y_{u,t}F_{2u}^{u-t}E[k(u,F_u,t)]\,du$$
Particularities of the path-dependent case

- Optimal surrender behaviour depends on account value $F_t$ and geometric average $Y_t$.
  - $\Rightarrow$ Optimal surrender surface

- To solve for optimal surrender surface, need to consider many values $F_t$ at each time $t$.

- To simplify calculations, assume that $B_t(F_t)$ has the form

$$B_t(F_t) = \max(G_T e^{-r(T-t)}, a_t + b_t F_t)$$
Numerical example

Additional assumptions:

- $T = 10$
- Payoff: $\max(Y_T, G_T)$
- $G_T = F_0 e^{0.025T}$
- $c = 0.0197$
Conclusion

- Integral representation for the surrender option
- Can retrieve optimal surrender boundary
- Can be used for path independent and path dependent payoffs

Future work:
- Use for other types of fee structures
- Consider flexible premium (as in Chi and Lin (2013))


Thank you for your attention!