Derivative pricing with non-Gaussian GARCH models and their continuous time limits

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3rd Workshop on Insurance Mathematics, Québec, January 31, 2014
Outline

1. Modelling the underlying under physical measure
2. Modelling the underlying under risk-neutral measures
3. Continuous time diffusion limits
4. Numerical Experiments
NGARCH(1,1) approximations under the physical measure

- $\mathcal{T}^{(n)} = \{ l | l = k/n, k = 0, 1 \ldots, nT \}$ set of trading dates. The length of each subinterval is $\tau = 1/n$.

- $\left\{ Y_l^{(n)} \right\}_{l \in \mathcal{T}^{(n)}} = \left\{ \log S_l^{(n)} \right\}_{l \in \mathcal{T}^{(n)}}$ has the following dynamic:

\[
Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} = \left( r + \lambda^{(e)} \sqrt{h_{k\tau}^{(n)}} - \kappa^{(n)}_{\epsilon k\tau} \left( \sqrt{h_{k\tau}^{(n)}} \right) \right) \tau + \sqrt{\tau} h_{k\tau}^{(n)} \epsilon_{k\tau}^{(n)},
\]

(1)

\[
h_{k\tau}^{(n)} - h_{(k-1)\tau}^{(n)} = \alpha_0(\tau) + \alpha_1(\tau) h_{(k-1)\tau}^{(n)} \left( \epsilon_{(k-1)\tau}^{(n)} - \gamma(\tau) \right)^2 + (\beta_1(\tau) - 1) h_{(k-1)\tau}^{(n)},
\]

(2)

\[
\epsilon_{k\tau}^{(n)} | \mathcal{F}_{(k-1)\tau}^{(n)} \sim D(0, 1).
\]

(3)

- These are defined on $\left( \Omega^{(n)}, \mathcal{F}^{(n)}, \left\{ \mathcal{F}_l^{(n)} \right\}_{l \in \mathcal{T}^{(n)}}, P^{(n)} \right)$ with

\[
\mathcal{F}_{k\tau}^{(n)} = \sigma \left( Y_0^{(n)}, Y_1^{(n)}, \ldots, Y_{k\tau}^{(n)} \right).
\]

- $\left\{ \epsilon_{k\tau}^{(n)} \right\}_{k=0, \ldots, nT}$ is a sequence of $\mathcal{F}_{(k-1)\tau}^{(n)}$ - conditionally i.i.d. with zero mean and unit variance distribution $D$. 

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Modelling the underlying under the physical measure

- \( f_{\epsilon_{k\tau}}(\cdot) \) the p.d.f. and \( \kappa_{\epsilon_{k\tau}}(\cdot) \) the c.g.f. of \( \epsilon_{k\tau} \) conditional on \( \mathcal{F}_{(k-1)\tau} \) under \( P^{(n)} \)

\[
\kappa_{\epsilon_{k\tau}}^{(n)}(u) := \ln \mathbb{E}^{P^{(n)}} \left[ \exp\left( u\epsilon_{k\tau}^{(n)} \right) \bigg| \mathcal{F}_{(k-1)\tau}^{(n)} \right] < \infty.
\]

- The innovations’ \( j^{th} \) raw moments are:

\[
M_j = \mathbb{E}^{P^{(n)}} \left[ \left( \epsilon_{k\tau}^{(n)} \right)^j \bigg| \mathcal{F}_{(k-1)\tau}^{(n)} \right].
\]

- The parameter \( \lambda^{(\epsilon)} \) is usually interpreted as the market price of \( \epsilon \) risk because we have:

\[
\mathbb{E}^{P^{(n)}} \left[ \exp\left( Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} \right) \bigg| \mathcal{F}_{(k-1)\tau}^{(n)} \right] = \exp \left( \tau \left( r + \lambda^{(\epsilon)} \sqrt{h_{k\tau}^{(n)}} \right) \right).
\]

- When \( \tau = 1 \) the model (1)-(3) reduces to a general asymmetric NGARCH(1,1) model.
Identify a change of measure such that discounted asset prices follow the distribution of their martingale component in:

\[
\tilde{S}_{k\tau}^{(n)} = \tilde{S}_0^{(n)} A_{k\tau}^{(n)} M_{k\tau}^{(n)}.
\]

- \(M_{k\tau}^{(n)}\) is a martingale under \(P^{(n)}\) and \(W_{k\tau}^{(n)} := M_{k\tau}^{(n)} / M_{(k-1)\tau}^{(n)}\).

- \(\nu_{k\tau}^{(n)}\) is the one period excess discounted return process:

\[
\nu_{k\tau}^{(n)} = -r\tau + \log \mathbb{E}^{P^{(n)}} \left[ \exp \left( Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} \right) \bigg| \mathcal{F}_{(k-1)\tau}^{(n)} \right].
\]

The Radon-Nikodym process \(Z_{k\tau}^{(n)}\) is defined via the conditional p.d.f. \(g_{W_{k\tau}^{(n)}}(\cdot)\) of \(W_{k\tau}^{(n)}\) under \(P^{(n)}\):

\[
Z_{k\tau}^{(n)} := \frac{dQ^{(n)}_{egp}}{dP^{(n)}} \bigg| \mathcal{F}_{k\tau}^{(n)} = \prod_{l=1}^{k} g_{W_{l\tau}^{(n)}}^{(n)} \left( \frac{\tilde{S}_{l\tau}^{(n)}}{\tilde{S}_{(l-1)\tau}^{(n)}} \right) e^{\nu_{l\tau}^{(n)}}
\]
Risk-neutralized dynamics under EGP

Proposition

The risk neutral dynamics of the process \( \{ Y_{(n)}^{(k)}(\tau), h^{(n)}_{(k)}(\tau) \}_{k=0, \ldots, nT} \) introduced in (1)-(3) with respect to the extended Girsanov risk neutral measure \( Q_{\text{egp}}^{(n)} \) are:

\[
Y_{(n)}^{(k)}(\tau) - Y_{(k-1)}^{(n)}(\tau) = \left( r - \frac{1}{\tau} \kappa_{(n)}^{(k)} \left( \sqrt{\tau h^{(n)}_{(k)}} \right) \right) \tau + \sqrt{\tau h^{(n)}_{(k)}} e^{*}_{(n)}(\tau),
\]

\[
h^{(n)}_{(k)}(\tau) - h^{(n)}_{(k-1)}(\tau) = \alpha_{0}(\tau) + \alpha_{1}(\tau) h^{(n)}_{(k-1)}(\tau) \left( e^{*}_{(k-1)}(\tau) - \sqrt{\tau q^{(n)}_{(k-1)}} - \gamma(\tau) \right)^2 + (\beta_{1}(\tau) - 1) h^{(n)}_{(k-1)}(\tau),
\]

\[
e^{*}_{(n)}(\tau) \mid \mathcal{F}_{(k-1)}^{(n)} \sim \mathcal{D}(0,1).
\]

Here, the innovation process \( \{ e^{*}_{(n)}(\tau) \}_{k=0, \ldots, nT} \) is a sequence of \( \mathcal{F}_{(k-1)}^{(n)} \)-conditionally uncorrelated zero mean and unit variance \( \mathcal{D} \)-distributed random variables under \( Q_{\text{egp}}^{(n)} \), related to the original innovations via the expression:

\[
e^{*}_{(n)}(\tau) = e^{(n)}_{(k)}(\tau) + \sqrt{\tau q^{(n)}_{(k)}}(\tau), \quad k = 0, \ldots, nT,
\]

where \( q^{(n)}_{(k)}(\tau) \) is given by:

\[
q^{(n)}_{(k)}(\tau) = \lambda(\varepsilon) + \frac{1}{\tau} \kappa_{(n)}^{(k)} \left( \sqrt{\tau h^{(n)}_{(k)}} \right) - \kappa_{(n)}^{(k)} \left( \sqrt{h^{(n)}_{(k)}} \right).\]
Conditional Esscher transform (ESS)

- Define the stochastic process $Z^{(n)} = \{Z_{kT}^{(n)}\}_{k=0,...,nT}$:
  
  $$Z_{kT}^{(n)} = \prod_{l=1}^{k} e^{-\sqrt{T}\theta^{(n)}_{lT}e^{(n)}_{lT} - \kappa^{(n)}_{lT}(\sqrt{T}\theta^{(n)}_{lT})}$$
  
  $Z_{0,n} = 1$.

- $\theta^{(n)} = \{\theta^{(n)}_{kT}\}_{k=0,...,nT}$ is an $\mathcal{F}^{(n)}$ predictable process satisfying:
  
  $$\mu_{kT}^{(n)} + \frac{1}{T}\kappa^{(n)}_{\epsilon_{kT}} \left(\sqrt{T} \left(\sqrt{h_{kT}^{(n)}} - \theta_{kT}^{(n)}\right)\right) - \frac{1}{T}\kappa^{(n)}_{\epsilon_{kT}} (\sqrt{T}\theta_{kT}^{(n)}) = r.$$

- Here, $\mu_{kT}^{(n)} = r + \lambda(\epsilon) \sqrt{h_{kT}^{(n)}} - \kappa^{(n)}_{\epsilon_{kT}} \left(\sqrt{h_{kT}^{(n)}}\right)$.

- $Z^{(n)}$ is a $P^{(n)}$ martingale and $Z_{nT}^{(n)}$ defines the equivalent martingale measure $Q^{(n)}_{ess}$ by $Z_{nT}^{(n)} = \frac{dQ^{(n)}_{ess}}{dP^{(n)}}$. 
Risk-neutralized dynamics under ESS

Proposition

The risk neutral dynamics of \( \{ Y_{k\tau}^{(n)}, h_{k\tau}^{(n)} \}_{k=0, \ldots, nT} \) under the exponential affine pricing kernel \( Q_{\text{ess}}^{(n)} \) are:

\[
Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} = \left( r + \frac{1}{\sqrt{\tau}} \sqrt{h_{k\tau}^{(n)} \kappa'_{\epsilon_k^{(n)}} (-\sqrt{\tau} \theta_{k\tau}^{(n)})} \right) \tau \\
+ \kappa_{\epsilon_k^{(n)}} ( -\sqrt{\tau} \theta_{k\tau}^{(n)} ) - \kappa_{\epsilon_k^{(n)}} \left( \sqrt{\tau} \left( \sqrt{h_{k\tau}^{(n)} - \theta_{k\tau}^{(n)}} \right) \right) \\
+ \sqrt{\tau} h_{k\tau}^{(n)} \sqrt{\kappa'_{\epsilon_k^{(n)}} ( -\sqrt{\tau} \theta_{k\tau}^{(n)})} \epsilon^{*}_{(n)}
\]

\[
h_{k\tau}^{(n)} - h_{(k-1)\tau}^{(n)} = \alpha_0 (\tau) + \alpha_1 (\tau) h_{k\tau}^{(n)} \left( \sqrt{\kappa'_{\epsilon_k^{(n)}} ( -\sqrt{\tau} \theta_{k\tau}^{(n)})} \epsilon^{*}_{k\tau} + \kappa_{\epsilon_k^{(n)}} \left( -\sqrt{\tau} \theta_{k\tau}^{(n)} \right) - \gamma (\tau) \right)^2 \\
+ (\beta_1 (\tau) - 1) h_{k\tau}^{(n)} \\
\epsilon^{*}_{k\tau} | \mathcal{F}_{(k-1)\tau} \sim D^* (0, 1).
\]

The innovation \( \{ \epsilon^{*}_{k\tau} \}_{k=0, \ldots, nT} \) is a sequence of \( \mathcal{F}_{(k-1)\tau} \)-conditionally \( D^* (0, 1) \)-distributed under \( Q_{\text{ess}}^{(n)} \):

\[
\epsilon^{*}_{k\tau} = \frac{\epsilon_{k\tau}^{(n)} - \kappa'_{\epsilon_k^{(n)}} \left( -\sqrt{\tau} \theta_{k\tau}^{(n)} \right)}{\sqrt{\kappa'_{\epsilon_k^{(n)}} ( -\sqrt{\tau} \theta_{k\tau}^{(n)})}}.
\]
Gaussian Innovations

- When innovations are Gaussian distributed, both propositions leads to the same risk neutralized dynamic obtained via the local risk neutral valuation principle (LRNVP).

\[ \varrho_{k\tau}^{(n)} = \theta_{k\tau}^{(n)} = \lambda^{(\epsilon)} \] for any \( k = 0, \ldots, nT \).

- When \( \tau = 1 \), these equations reduces to the GARCH option pricing of Duan (1995) with a leverage effect.

- The Radon-Nykodym derivative has the following explicit form of a discretized Girsanov change of measure in continuous time corresponding to a market price of risk \( \lambda^{(\epsilon)} \):

\[
\left. \frac{dQ^{(n)}}{dP^{(n)}} \right|_{\mathcal{F}_{k\tau}^{(n)}} = \exp\left( \sum_{l=1}^{k} \left( -\sqrt{\tau} \lambda^{(\epsilon)} \epsilon_{k\tau}^{(n)} - \frac{1}{2} \tau \left( \lambda^{(\epsilon)} \right)^2 \right) \right).
\]
Literature review

- Nelson (1990)
- Alexander and Lazar (2005)
- Corradi (2000)
- Lindner (2009)
- Duan et al. (2009) and Stentoft (2011)
Construction and constraints

- The right continuous with left limit (cadlag) extension of the proposed discrete time process is defined by:
  \[
  \left\{ Y_t^{(n)}, h_t^{(n)} \right\}_{k\tau \leq t < (k+1)\tau} := \left\{ Y_{k\tau}^{(n)}, h_{k\tau}^{(n)} \right\}, \quad k = 0, \ldots, nT.
  \]

- Define \( \mathcal{F}_{t}^{(n)} \) as:
  \[
  \mathcal{F}_{t}^{(n)} := \mathcal{F}_{k\tau}^{(n)}, \quad k = 0, \ldots, nT, \text{ and we denote by }
  \mathcal{F}_{t}^{(n),h} := \mathcal{F}_{t}^{(n)} \bigcup \left\{ h_t^{(n)} = h \right\}.
  \]

- **Parametric Constraints**
  \[
  \lim_{\tau \to 0} \frac{\alpha_0(\tau)}{\tau} = \omega_0, \quad \lim_{\tau \to 0} \frac{\alpha_1(\tau)(1 + \gamma^2(\tau)) + \beta_1(\tau) - 1}{\tau} = -\omega_1,
  \]
  \[
  \lim_{\tau \to 0} \frac{\alpha_1^2(\tau)}{\tau} = \omega_2, \quad \lim_{\tau \to 0} \gamma(\tau) = \omega_3.
  \]
Stochastic volatility limit under $P$

Proposition

Assume the above parameter conditions hold. Then, as $\tau$ approaches zero, the process $\{Y^{(n)}_t, h^{(n)}_t\}$ converges weakly to a bivariate diffusion $\left( Y_t, \sigma^2_t \right)$ which satisfies the following stochastic differential equation:

\[
\begin{align*}
    dY_t &= \left( r + \lambda(\epsilon) \sqrt{h_t} - \kappa_t \left( \sqrt{h_t} \right) \right) dt + \sqrt{h_t} dB_t^{(1)}, \\
    dh_t &= (\omega_0 - \omega_1 h_t) dt + \sqrt{\omega_2}(M_3 - 2\omega_3) h_t dB_t^{(1)} \\
    &\quad + \sqrt{\omega_2} \sqrt{M_4 - M_3^2 - 1} h_t dB_t^{(2)}.
\end{align*}
\]

Here, $B_t^{(1)}$ and $B_t^{(2)}$ are two independent Brownian motions on $\left( \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \ldots, T]}, P \right)$.

- In the case of Gaussian innovations this coincides with the standard asymmetric GARCH diffusion limit of Duan (1997).
- The use of a non-Gaussian distribution for the underlying discrete process does not alter the Hull-White structure of the variance equation.
- The diffusion coefficient of the stochastic volatility dynamics incorporates the skewness and the kurtosis of the distribution.
Stochastic volatility limit under $Q$

**Proposition**

Under the same parametric conditions, the risk neutral processes under $Q_{egp}^{(n)}$ and $Q_{ess}^{(n)}$ converge weakly to the same bivariate diffusion limit given below:

$$dY_t = \left( r - \frac{1}{2} h_t \right) dt + \sqrt{h_t} dB^{*(1)}_t,$$

$$dh_t = \left( \omega_0 - \left( \omega_1 + \sqrt{\omega_2(M_3 - 2\omega_3)} \nu^{(1)}_t + \sqrt{\omega_2} \nu^{(2)}_t \sqrt{M_4 - M_3^2 - 1} \right) h_t \right) dt$$

$$+ \sqrt{\omega_2} (M_3 - 2\omega_3) h_t dB^{*(1)}_t + \sqrt{\omega_2} \sqrt{M_4 - M_3^2 - 1} h_t dB^{*(2)}_t.$$

Here $B^{*(1)}_t$ and $B^{*(2)}_t$ are two independent Brownian motions under $Q$:

$$B^{*(1)}_t = B^{(1)}_t + \int_0^t \nu^{(1)}_s ds, \quad B^{*(2)}_t = B^{(2)}_t + \int_0^t \nu^{(2)}_s ds,$$

and the market prices of $B^{*(1)}_t$ and $B^{*(2)}_t$ risk are given by:

$$\nu^{(1)}_t = \lambda(\epsilon) + \frac{1}{2} h_t - \kappa_t(\sqrt{h_t}), \quad \nu^{(2)}_t = -\nu^{(1)}_t \frac{M_3}{\sqrt{M_4 - M_3^2 - 1}}.$$
Comments

- This can be viewed as an extension of Duan’s (1996) convergence theorem of locally risk neutralized Gaussian GARCH models obtained via LRNVR.

- When the driving noise is Gaussian, the price of risk processes are $\nu_t^{(1)} = \lambda^{(c)}$ and $\nu_t^{(2)} = 0$ (since $M_3 = 0$).

- The variance equation reduces to the well-known GARCH diffusion process which is obtained by applying the minimal martingale measure to the SV model under $P$.

$$dh_t = (\omega_0 - (\omega_1 - 2\sqrt{\omega_2\omega_3}) h_t) \, dt - 2\sqrt{\omega_2\omega_3} h_t dB_t^{(1)} + \sqrt{2\omega_2} h_t dB_t^{(2)}.$$  

- Moreover, since $\nu_t^{(2)} = 0$ whenever $M_3 = 0$, the minimal martingale measure is obtained as the weak limit for any symmetric distribution used for modelling the underlying GARCH process.

- Skewness plays an important role here as it induces a non-zero market price of non-hedgeable risk in the continuous time limit.
• The resulting variance process does not have a Hull-White structure since the drift is not a linear function of $h_t$.

• It has a non-linear dependence through the cumulant generating function of the GARCH noise.

• Using a Taylor expansion of the second order or of the fourth order we obtain:

$$\nu^{(1)}_t = \lambda(\epsilon), \quad \nu^{(1)}_t = \lambda(\epsilon) - \frac{1}{6} M_3 h_t - \frac{1}{24} \sqrt{h_t^3} (M_4 - 3).$$

• The resulting return equation ensures that the discounted asset price is a local martingale under $Q$.

• $\tilde{S}_t$ is a true martingale is equivalent to having a non-positive correlation between the asset return and its variance provided that the market price of $B_t^{(1)}$ risk $\nu^{(1)}_t$ is bounded.

$$\text{Cov}(dY_t, dh_t) = \sqrt{\omega_2} (M_3 - 2\omega_3) \sqrt{h_t^3}.$$
NGARCH(1,1) innovations, estimation and simulation

- We compute differences between NGARCH(1,1) based on Gaussian and NIG innovations and their diffusion limits option prices.

- For NIG we denote $\epsilon^{(n)}_{k\tau} \sim \text{NIG}(k, a, s, \ell)$, with cumulant generating function given by:

$$\kappa^{(n)}_{\epsilon_{k\tau}}(z) = z\ell + \left(\sqrt{k^2 - a^2} - \sqrt{k^2 - (a + zs)^2}\right).$$

- The model parameters are obtained by fitting the NGARCH(1,1) model from (1)-(2) with $\tau = 1$ via maximum likelihood estimation (MLE). We use daily log-returns of the S&P 500 index from January 2nd, 1988 to April 17th, 2002.

  - **NGARCH(1,1) model with Gaussian innovations (NGARCH):**

    $\alpha_0(1) = 9.9411 \cdot 10^{-7}$, $\alpha_1(1) = 0.0417$, $\beta_1(1) = 0.9176$, $\gamma(1) = 0.8639$, $\lambda = 0.0414$.

  - **NGARCH(1,1) model with NIG innovations (NIG-NGARCH):**

    $\alpha_0(1) = 8.6650 \cdot 10^{-7}$, $\alpha_1(1) = 0.0479$, $\beta_1(1) = 0.9096$, $\gamma(1) = 0.8601$, $\lambda = 0.0419$.

  - Additionally, the NIG invariant parameters are $k = 1.7190$ and $a = -0.1869$. 
• We compute the prices associated with European put options based on the daily GARCH process as well as its diffusion limits.

• GARCH prices are computed as discounted expected payoffs under the risk neutral measure, and since there are no closed form solutions

• Since there are no closed form solutions, prices are obtained as the average of 120 Monte Carlo simulations involving 100,000 paths each.

• Diffusion prices are computed based on sample paths simulated using an Euler discretization of 1,024 steps per day, and the model parameters are those induced by the corresponding GARCH processes.

• We use the empirical martingale simulation (EMS) of Duan and Simonato (1998).
## Gaussian NGARCH and its diffusion limit Put option prices

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### Numerical Experiment

#### NIG - NGARCH and its diffusion limit Put option prices

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<td>(0.007)</td>
<td>(0.009)</td>
<td>(0.004)</td>
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<td>4.783</td>
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<td>(0.010)</td>
<td>(0.014)</td>
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<td>NIG-NGARCH</td>
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<td>(0.021)</td>
<td>(0.016)</td>
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Convergence of Gaussian GARCH option prices to SV

Figure: Convergence of the Gaussian GARCH option prices to their continuous time limit counterparts.
Figure: Convergence of the NIG-GARCH option prices to their continuous time limit counterparts.
Pricing errors for SV models based on MMM and EGP

Figure: Square differences between NIG-GARCH diffusion limits prices based and MMM and EGP
### NIG - NGARCH and its diffusion limit Call option prices

<table>
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<tr>
<th>Maturity</th>
<th>Model</th>
<th>K=450</th>
<th>K=470</th>
<th>K=490</th>
<th>K=510</th>
<th>K=530</th>
<th>K=550</th>
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<tr>
<td>7</td>
<td>NIG-NGARCH</td>
<td>50.116 (0.036)</td>
<td>30.359 (0.042)</td>
<td>12.392 (0.035)</td>
<td>2.034 (0.015)</td>
<td>0.109 (0.003)</td>
<td>0.005 (0.001)</td>
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<td>SV</td>
<td>50.090 (0.046)</td>
<td>30.289 (0.044)</td>
<td>12.424 (0.036)</td>
<td>2.088 (0.014)</td>
<td>0.064 (0.002)</td>
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<tr>
<td>21</td>
<td>NIG-NGARCH</td>
<td>50.880 (0.051)</td>
<td>32.284 (0.051)</td>
<td>16.347 (0.043)</td>
<td>5.714 (0.028)</td>
<td>1.223 (0.012)</td>
<td>0.173 (0.005)</td>
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<tr>
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<td>SV</td>
<td>50.807 (0.074)</td>
<td>32.261 (0.067)</td>
<td>16.455 (0.053)</td>
<td>5.808 (0.031)</td>
<td>1.154 (0.012)</td>
<td>0.105 (0.003)</td>
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<td>54.903 (0.075)</td>
<td>38.260 (0.069)</td>
<td>23.964 (0.058)</td>
<td>12.987 (0.046)</td>
<td>5.868 (0.033)</td>
<td>2.178 (0.021)</td>
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<td>38.267 (0.106)</td>
<td>24.011 (0.086)</td>
<td>13.005 (0.065)</td>
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<tr>
<td>126</td>
<td>NIG-NGARCH</td>
<td>60.865 (0.088)</td>
<td>45.412 (0.081)</td>
<td>31.873 (0.072)</td>
<td>20.743 (0.062)</td>
<td>12.347 (0.049)</td>
<td>6.657 (0.035)</td>
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<td>SV</td>
<td>60.674 (0.142)</td>
<td>45.225 (0.126)</td>
<td>31.684 (0.107)</td>
<td>20.536 (0.088)</td>
<td>12.108 (0.066)</td>
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<td>70.954 (0.106)</td>
<td>56.644 (0.100)</td>
<td>43.825 (0.091)</td>
<td>32.721 (0.081)</td>
<td>23.481 (0.073)</td>
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<tr>
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<td>SV</td>
<td>70.297 (0.187)</td>
<td>55.952 (0.171)</td>
<td>43.120 (0.153)</td>
<td>32.023 (0.133)</td>
<td>22.812 (0.111)</td>
<td>15.523 (0.091)</td>
</tr>
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</table>
Hedging with GARCH models

Proposition

Let \( H \) be a European claim. Let
\[
\Pi_{Q_n}^{\min}(n) \min_{\tau} = \Pi_{Q_n}^{\min}(S_{k\tau}, h_{(k+1)\tau}) = E^{Q_n}_{\min}\left[e^{r(T-t)}H \mid \mathcal{F}_{k\tau}\right]
\]
and
\[
\Pi_{Q_n}^{(n)}(S_{k\tau}, h_{(k+1)\tau}) = E^{Q_n}\left[e^{r(T-t)}H \mid \mathcal{F}_{k\tau}\right]
\]
be associated prices based on \( Q_n^{\min} \) and another arbitrary pricing measure \( Q_n \), respectively.

The local risk minimization hedge under \( P_n \) can be approximated by:

\[
\xi_{P_n}(k+1\tau) = \Delta_{Q_n}^{\min} + VM_{(k+1)\tau}^{P_n} \Delta_{Q_n}^{\min},
\]

\[
VM_{(k+1)\tau}^{P_n} = \frac{\text{Cov}_{P_n}(S_{(k+1)\tau} - S_{k\tau}, h_{(k+2)\tau} - h_{(k+1)\tau} \mid \mathcal{F}_{k\tau})}{\text{Var}_{P_n}\left[S_{(k+1)\tau} - S_{k\tau} \mid \mathcal{F}_{k\tau}\right]}.
\]

The local risk minimization hedge under \( Q_n \) can be approximated by:

\[
\xi_{Q_n}(k+1\tau) = \Delta_{Q_n}^{\min} + VM_{(k+1)\tau}^{Q_n} \Delta_{Q_n}^{\min},
\]

\[
VM_{(k+1)\tau}^{Q_n} = \frac{\text{Cov}_{Q_n}(S_{(k+1)\tau} - S_{k\tau}, h_{(k+2)\tau} - h_{(k+1)\tau} \mid \mathcal{F}_{k\tau})}{\text{Var}_{Q_n}\left[S_{(k+1)\tau} - S_{k\tau} \mid \mathcal{F}_{k\tau}\right]}.
\]
Proposition

Suppose that the asset returns are governed by the non Gaussian GARCH stochastic volatility process. Then the locally risk minimizing hedging strategies with respect to $P$ and $Q$ (either EGP or ESS) are given by:

$$
\xi^P_t = \Delta_{t,S}^{Q_{min}} + \frac{\sqrt{h_t}}{S_t} \sqrt{\omega_2} (M_3 - 2\omega_3) \Delta_{t,h}^{Q_{min}},
$$

$$
\xi^Q_t = \Delta_{t,S}^Q + \frac{\sqrt{h_t}}{S_t} \sqrt{\omega_2} (M_3 - 2\omega_3) \Delta_{t,h}^Q.
$$