Extreme-Aggregation and Superadditivity

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Joint work with Valeria Bignozzi and Andreas Tsanakas

Outline

Basel Documents

From Basel Committee on Banking Supervision:

R1: Consultative Document, May 2012,
Fundamental review of the trading book

R2: Consultative Document, October 2013,
Fundamental review of the trading book: A revised market risk framework.

Basel Question

R1, Page 41, Question 8:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

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- Gneiting (2011): ES is not elicitable, while VaR is.

A feast for financial mathematicians and financial statisticians! See Embrechts et al (2014).

VaR and ES

Definition

$\mathrm{VaR}_p(X)$, for $p \in (0,1)$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Definition

$\mathrm{ES}_p(X)$, for $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_\delta(X)\mathrm{d}\delta \underset{(F\text{ cont.})}{=} \mathbb{E}\left[X|X > \mathrm{VaR}_p(X)\right].$$

VaR versus ES: Summary

Value-at-Risk

1. Always exists
2. Only frequency
3. Non-coherent risk measure (diversification problem)
4. Backtesting straightforward
5. Estimation: far in the tail
6. Model uncertainty: sensitive to dependence
7. Robust with respect to weak topology

Expected Shortfall

1. Needs first moment
2. Frequency and severity
3. Coherent risk measure (diversification benefit)
4. Backtesting an issue (non-elicitability)
5. Estimation: data limitation
6. Model uncertainty: sensitive to tail modeling
7. Robust with respect to Wasserstein distance

The Holy Triangle of Risk Measures

There are many desired properties of a good risk measure. Some properties are without debate:

- cash-invariance: $\rho(X + c) = \rho(X) + c$, $c \in \mathbb{R}$;
- monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$;
- zero-normalization: $\rho(0) = 0$;
- law-invariance: $\rho(X) = \rho(Y)$ if $X =_d Y$.

(A standard risk measure; those properties are not restrictive)

Another one is listed here as debatable:

- positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$, $\lambda \geq 0$.

The Holy Triangle of Risk Measures

In my opinion, in addition to being standard, the three key elements of being a good risk measure are

(C) Coherence (subadditivity): $\rho(X+Y) \leq \rho(X) + \rho(Y)$. [diversification benefit/aggregate regulation/capturing the tail]

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(A) Comonotone additivity: $\rho(X+Y) = \rho(X) + \rho(Y)$ if X and Y are comonotone. [economical interpretation/distortion representation/non-diversification identity]

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(E) Elicitability [estimation advantage/backtesting straightforward].

The War of the Two Kingdoms

- Financial mathematicians
  - appreciate coherence (subadditivity);
  - favor ES in general.
- Financial statisticians
  - appreciate backtesting and statistical advantages;
  - favor VaR in general.

A natural question is to find a standard risk measure which is both coherent (subadditive) and elicitable.

Expectiles

Expectiles

For $0 < \tau < 1$ and $X \in L^2$, the p-expectile is

$$e_p(X) = \operatorname*{argmin}_{x \in \mathbb{R}} \mathbb{E}[p(X-x)_+^2 + (1-p)(x-X)_+^2].$$

- $e_p(X)$ is the unique solution x of the equation for $X \in L^1$:

$$p\mathbb{E}[(X-x)_+] = (1-p)\mathbb{E}[(x-X)_+].$$

- $e_{1/2}(X) = \mathbb{E}[X]$.
- If we allow $p = 1$: $e_1(X) = \text{ess-sup}(X)$.

Expectiles

The risk measure e_p has the following properties:

1. positive homogeneous and standard;
2. subadditive for $1/2 \leq p < 1$, superadditive for $0 < p \leq 1/2$;
3. elicitable;
4. coherent for $1/2 \leq p < 1$;
5. not comonotonic additive in general.

Bellini et al. (2014), Ziegel (2014), Delbaen (2014).

The War of the Three Kingdoms

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In summary:

- VaR has (A) and (E): often criticized for not being subadditive: diversification/aggregation problems and inability to capture the tail!
- ES has (C) and (A): criticized for estimation, backtesting and robustness problems!
- Expectile has (C) and (E): criticized for lack of economical meaning, distributional computation and over-diversification benefits!

The War of the Three Kingdoms

The following hold:

- if ρ is coherent, comonotonic additive and elicitable, then ρ is the mean (Ziegel, 2014);
- if ρ is coherent, and elicitable with a convex scoring function, then ρ is an expectile (Bellini and Bignozzi, 2014);

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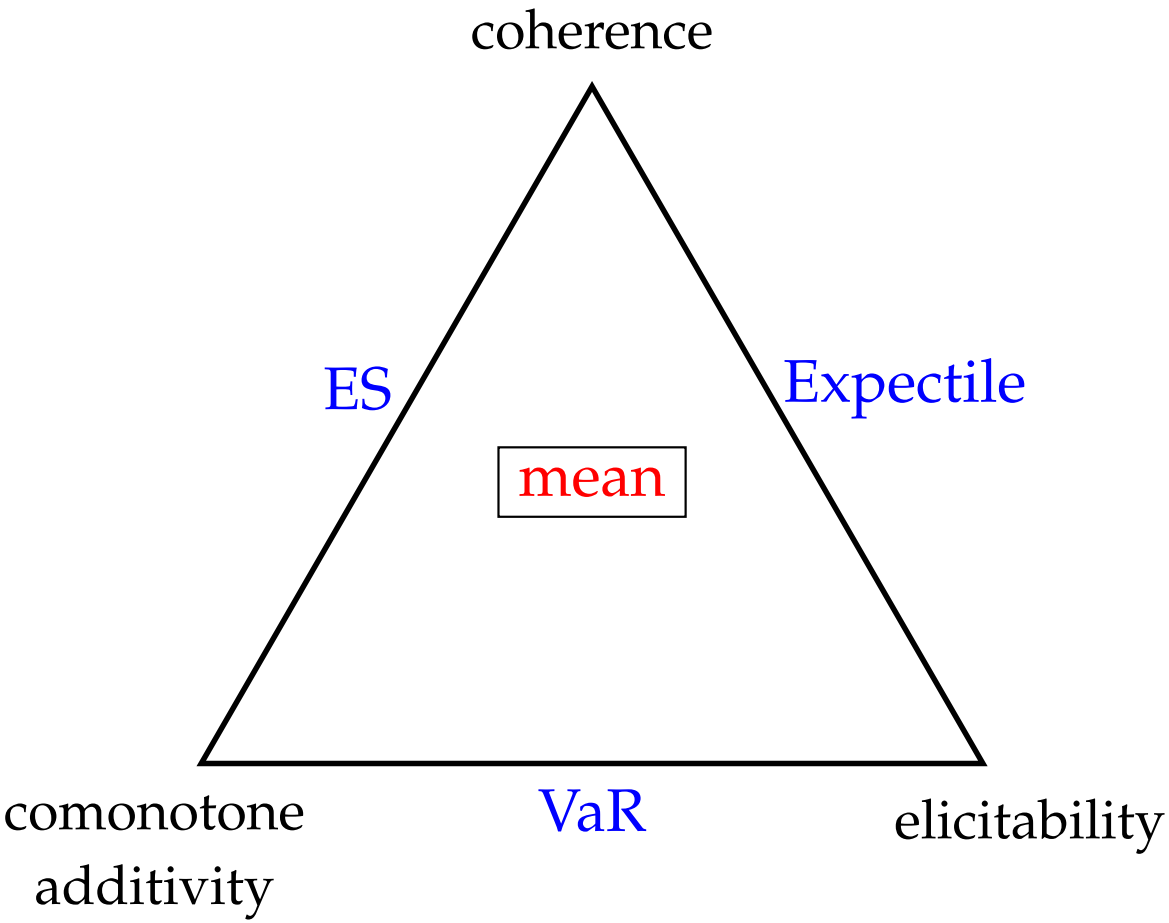
- Remark: the very old-school risk measure/pricing principle $\rho(X) = (1+\theta)\mathbb{E}[X]$, $\theta > 0$ has (C-subadditivity), (A) and (E), although it is not standard.

The Holy Triangle of Risk Measures



Subadditivity

Subadditivity has to do with

- diversification benefit - *"a merger does not create extra risk"*.
- aggregation - manipulation of risk: $X \to Y + Z$;
- capturing the tail.

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It is questioned from different aspects:

- aggregation penalty - convex risk measures;
- robustness and backtesting;
- financial practice - *"a merger creates extra risk"*.

How Superadditive Can a Risk Measure be?

Question: given an non-subadditive risk measure,

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Motivation:

- Measure model uncertainty.
- Quantify worst-scenarios.
- Trade subadditivity for statistical advantages such as robustness or elicitability.
- Understand better about coherence.

Diversification ratio

For a law-invariant (always assumed) risk measure ρ, and risks $\mathbf{X} = (X_1, \cdots, X_n)$, the diversification ratio is defined as

$$\Delta^{\mathbf{X}}(\rho) = \frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)}.$$

For the moment, the denominator is assumed positive.

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For the moment, the denominator is assumed positive.

- $\Delta^{\mathbf{X}}(\rho)$ is important in modeling portfolios.
- We want to know how large $\Delta^{\mathbf{X}}(\rho)$ can be.
- $\Delta^{\mathbf{X}}(\rho) \leq 1$ for subadditive risk measures.
- We cannot take a supremum over all possible $\mathbf{X}$, which often explodes for any non-superadditive risk measure.

Diversification ratio

We define a law-invariant version of diversification ratio:

$$\Delta_n^F(\rho) = \sup\left\{\frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)} : X_1, \ldots, X_n \sim F\right\}.$$

Here we assumed homogeneity in F_i for:

- mathematical tractability;
- that it makes sense to let n vary;
- that it characterizes the superadditivity of ρ (in some sense) eventually.

Diversification ratio

Define

$$\mathfrak{S}_n(F) = \{X_1 + \cdots + X_n : X_1, \ldots, X_n \sim F\}.$$

Let $X_F \sim F$. Then

$$\Delta_n^F(\rho) = \frac{1}{n\rho(X_F)} \sup \{\rho(S) : S \in \mathfrak{S}_n(F)\}.$$

- Known to be a difficult problem; explicit solution for VaR (under some strong conditions) given in W., Peng and Yang (2013).
- Numerical calculation for VaR given in Embrechts, Puccetti and Rüschendorf (2013).

Extreme-aggregation measure

We are interested in the global superadditivity ratio

$$\Delta^F(\rho) = \sup_{n\in\mathbb{N}} \Delta_n^F(\rho) = \sup_{n\in\mathbb{N}} \frac{1}{n\rho(X_F)} \sup\left\{\rho(S) : S \in \mathfrak{S}_n(F)\right\}.$$

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The real mathematical target:

$$\limsup_{n\to\infty} \frac{1}{n} \sup\left\{\rho(S) : S \in \mathfrak{S}_n(F)\right\}.$$

Extreme-aggregation measure

Definition

An extreme-aggregation measure induced by a law-invariant risk measure ρ is defined as

$$\Gamma_\rho : L^0 \to [-\infty, \infty], \quad \Gamma_\rho(X_F) = \limsup_{n\to\infty} \frac{1}{n} \sup \{\rho(S) : S \in \mathfrak{S}_n(F)\}.$$

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- Γ_ρ quantifies the limit of ρ for worst-case aggregation under dependence uncertainty.
- Γ_ρ is a law-invariant risk measure.

Extreme-aggregation measure

Proposition

If ρ is (i) positive homogeneous, (ii) comonotonic additive, or (iii) convex and zero-normal, then

$$\Gamma_\rho(X_F) = \sup_{n\in\mathbb{N}} \frac{1}{n} \sup\{\rho(S) : S \in \mathfrak{S}_n(F)\} \geq \rho(X_F)$$

If ρ is subadditive then $\Gamma_\rho \leq \rho$. If it also satisfies (i), (ii) or (iii), then $\Gamma_\rho = \rho$.

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Remark

Γ_ρ inherits monotonicity, cash-invariance, positive homogeneity, subadditivity, convexity, or zero-normalization from ρ if ρ has the corresponding properties.

Extreme-aggregation measure

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What is Γ_ρ?

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- Known motivating result (Wang and W., 2014): as $n \to \infty$,

$$\frac{\sup\{\mathrm{VaR}_p(S) : S \in \mathfrak{S}_n(F)\}}{\sup\{\mathrm{ES}_p(S) : S \in \mathfrak{S}_n(F)\}} \to 1.$$

This implies $\Gamma_{\mathrm{VaR}_p} = \Gamma_{\mathrm{ES}_p} = \mathrm{ES}_p$.

Main result for distortion risk measures

Distortion risk measures:

$$\rho(X_F) = \int_0^1 F^{-1}(t)\mathrm{d}h(t).$$

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Theorem (Extreme-aggregation for distortion risk measures)

Suppose ρ is a distortion risk measure with distortion function h, then Γ_ρ is

(a) *the smallest coherent risk measure dominating ρ;*

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- Example: $\Gamma_{\mathrm{VaR}_p} = \mathrm{ES}_p$.
- A proof of more than 10 pages (single spaced).

Main result for distortion risk measures

For distortion risk measures,

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Theorem (Coherence and extreme-aggregation)

Suppose ρ is distortion risk measure. The following are equivalent:

(a) *ρ is coherent.*

(b) *$\Gamma_\rho(X_F) = \rho(X_F)$ for all distributions F.*

(c) *$\Gamma_\rho(X_F) = \rho(X_F)$ for some continuous distribution F, $\rho(X_F) < \infty$.*

(d) *$\Delta^F(\rho) = 1$ for all distributions F, $\rho(X_F) \in (0, \infty)$.*

(e) *$\Delta^F(\rho) = 1$ for some continuous distribution F, $\rho(X_F) \in (0, \infty)$.*

Main result for shortfall risk measures

Shortfall risk measures:

$$\rho(X) = \inf\{y \in \mathbb{R} : \mathbb{E}[\ell(X - y)] \leq l(0)\}.$$

ℓ: convex and increasing function. A loss function.

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Suppose ρ is a shortfall risk measure with loss function ℓ, then Γ_ρ is

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(b) *a coherent p-expectile, where*

$$p = \lim_{x\to\infty} \ell'(x)/(\lim_{x\to\infty} \ell'(x) + \lim_{x\to-\infty} \ell'(x))$$

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- Example: $\Gamma_{\mathrm{ER}_\beta} = e_1 = \text{ess-sup}$, where ER_β is the entropy risk measure: with loss function $\ell(x) = \exp(\beta x) - 1$.
- A proof of one page.

Conclusion

- Γ_ρ is always a coherent risk measure in all studied examples whenever ρ is standard.
- Γ_ρ gains positive homogeneity, convexity, and subadditivity even if ρ does not have these properties, in all studied examples.
- A universal proof of this phenomenon is not available yet.
- Characterize the class of risk measures which induce coherent extreme-aggregation measures?

Conclusion

Some take-home message:

Coherence is indeed a natural property desired by a good risk measure. Even when a non-coherent risk measure is applied to a portfolio, its extreme behavior under dependence uncertainty leads to coherence.

This contributes to the Basel question and partly supports the use of coherent risk measures.

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Thank you for your kind attendance!

Basel Question

R1, Page 20, *Choice of risk metric*:

"... However, a number of weaknesses have been identified with VaR, including its inability to capture "tail risk". The Committee therefore believes it is necessary to consider alternative risk metrics that may overcome these weaknesses."

Basel Question

We focus on the mathematical and statistical aspects, avoiding discussion on practicalities and operational issues.

From R1, Page 3:

"The Committee recognises that moving to ES could entail certain operational challenges; nonetheless it believes that these are outweighed by the benefits of replacing VaR with a measure that better captures tail risk."

Basel Question

R2, Page 3, *Approach to risk management*:

"the Committee has its intention to pursue two key confirmed reforms outlined in the first consultative paper [May 2012]: Stressed calibration . . . Move from Value-at-Risk (VaR) to Expected Shortfall (ES)."