Extreme-Aggregation and Superadditivity

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3rd Workshop on Insurance Mathematics
Laval University Quebec City Jan 31, 2014

Joint work with Valeria Bignozzi and Andreas Tsanakas
Outline

1. VaR and ES
2. The Holy Triangle of Risk Measures
3. How Superadditive Can a Risk Measure Be?
4. Conclusion
5. References
From Basel Committee on Banking Supervision:

**R1:** Consultative Document, May 2012,  
Fundamental review of the trading book

**R2:** Consultative Document, October 2013,  
R1, Page 41, Question 8:

“What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?”
**Basel Question**

R1, Page 41, Question 8:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

- Cont, Deguest and Scandolo (2010): ES is not robust, while VaR is.
- Gneiting (2011): ES is not elicitable, while VaR is.
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**VaR and ES**

**Definition**

\( \text{VaR}_p(X) \), for \( p \in (0, 1) \),

\[
\text{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha \}.
\]

**Definition**

\( \text{ES}_p(X) \), for \( p \in (0, 1) \),

\[
\text{ES}_p(X) = \frac{1}{1 - p} \int_p^1 \text{VaR}_\delta(X) \, d\delta = F \left[ X \mid X > \text{VaR}_p(X) \right].
\]
### VaR versus ES: Summary

#### Value-at-Risk
1. **Always** exists
2. **Only** frequency
3. Non-coherent risk measure *(diversification problem)*
4. Backtesting straightforward
5. Estimation: far in the tail
6. Model uncertainty: sensitive to dependence
7. Robust with respect to weak topology

#### Expected Shortfall
1. **Needs** first moment
2. Frequency and **severity**
3. Coherent risk measure *(diversification benefit)*
4. Backtesting an issue *(non-elicitability)*
5. Estimation: data limitation
6. Model uncertainty: sensitive to tail modeling
7. Robust with respect to Wasserstein distance
The Holy Triangle of Risk Measures

There are many desired properties of a good risk measure. Some properties are without debate:

- cash-invariance: $\rho(X + c) = \rho(X) + c, c \in \mathbb{R}$;
- monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$;
- zero-normalization: $\rho(0) = 0$;
- law-invariance: $\rho(X) = \rho(Y)$ if $X \overset{d}{=} Y$.

(A standard risk measure; those properties are not restrictive)

Another one is listed here as debatable:

- positive homogeneity: $\rho(\lambda X) = \lambda \rho(X), \lambda \geq 0$. 

The Holy Triangle of Risk Measures

In my opinion, in addition to being standard, the three key elements of being a good risk measure are

(C) Coherence (subadditivity): $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

[diversification benefit/aggregate regulation/capturing the tail]
The Holy Triangle of Risk Measures

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(C) Coherence (subadditivity): \( \rho(X + Y) \leq \rho(X) + \rho(Y) \).
[diversification benefit/aggregate regulation/capturing the tail]

(A) Comonotone additivity: \( \rho(X + Y) = \rho(X) + \rho(Y) \) if \( X \) and \( Y \) are comonotone. [economical interpretation/distortion representation/non-diversification identity]
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(E) Elicitability [estimation advantage/backtesting straightforward].
The War of the Two Kingdoms

- Financial mathematicians
  - appreciate coherence (subadditivity);
  - favor ES in general.

- Financial statisticians
  - appreciate backtesting and statistical advantages;
  - favor VaR in general.

A natural question is to find a standard risk measure which is both coherent (subadditive) and elicitable.
Expectiles

For $0 < \tau < 1$ and $X \in L^2$, the $p$-expectile is

$$e_p(X) = \arg\min_{x \in \mathbb{R}} \mathbb{E}[p(X - x)^2_+ + (1 - p)(x - X)^2_+] .$$

- $e_p(X)$ is the unique solution $x$ of the equation for $X \in L^1$:

$$p\mathbb{E}[(X - x)_+] = (1 - p)\mathbb{E}[(x - X)_+] .$$

- $e_{1/2}(X) = \mathbb{E}[X]$.

- If we allow $p = 1$: $e_1(X) = \text{ess-sup}(X)$.  


Expectiles

The risk measure $e_p$ has the following properties:

1. positive homogeneous and standard;
2. **subadditive** for $1/2 \leq p < 1$, superadditive for $0 < p \leq 1/2$;
3. elicitable;
4. **coherent** for $1/2 \leq p < 1$;
5. **not comonotonic additive** in general.

In summary:

- VaR has (A) and (E): often criticized for not being subadditive: *diversification/aggregation problems and inability to capture the tail!*
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- ES has (C) and (A): criticized for *estimation, backtesting and robustness problems!*
The War of the Three Kingdoms

In summary:

- VaR has (A) and (E): often criticized for not being subadditive: diversification/aggregation problems and inability to capture the tail!

- ES has (C) and (A): criticized for estimation, backtesting and robustness problems!

- Expectile has (C) and (E): criticized for lack of economical meaning, distributional computation and over-diversification benefits!
The War of the Three Kingdoms

The following hold:

- if $\rho$ is coherent, comonotonic additive and elicitable, then $\rho$ is the mean (Ziegel, 2014);

- if $\rho$ is coherent, and elicitable with a convex scoring function, then $\rho$ is an expectile (Bellini and Bignozzi, 2014);
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In summary:

The only standard risk measure that has (C), (A) and (E) is the mean, which is not a tail risk measure, and does not have a risk loading.
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In summary:

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Remark: the very old-school risk measure/pricing principle $\rho(X) = (1 + \theta)\mathbb{E}[X], \theta > 0$ has (C-subadditivity), (A) and (E), although it is not standard.
The Holy Triangle of Risk Measures

- Coherence
- Comonotone additivity
- Elicitability
- Coherence
- VaR
- Mean

Risk Measures:
- VaR and ES
- Holy Triangle
- How Superadditive Can It Be?
- Conclusion
- References
Subadditivity has to do with

- diversification benefit - ""a merger does not create extra risk"".
- aggregation - manipulation of risk: $X \rightarrow Y + Z$;
- capturing the tail.
Subadditivity

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- diversification benefit - "a merger does not create extra risk".
- aggregation - manipulation of risk: $X \rightarrow Y + Z$;
- capturing the tail.

It is questioned from different aspects:

- aggregation penalty - convex risk measures;
- robustness and backtesting;
- financial practice - "a merger creates extra risk".
Question: given an non-subadditive risk measure, How superadditive can it be?
How Superadditive Can a Risk Measure be?

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How superadditive can it be?

Motivation:

- Measure model uncertainty.
- Quantify worst-scenarios.
- Trade subadditivity for statistical advantages such as robustness or elicitability.
- Understand better about coherence.
For a law-invariant (always assumed) risk measure $\rho$, and risks $X = (X_1, \ldots, X_n)$, the diversification ratio is defined as

$$\Delta^X(\rho) = \frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)}.$$ 

For the moment, the denominator is assumed positive.
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For the moment, the denominator is assumed positive.

- $\Delta^X(\rho)$ is important in modeling portfolios.
- We want to know how large $\Delta^X(\rho)$ can be.
- $\Delta^X(\rho) \leq 1$ for subadditive risk measures.
- We cannot take a supremum over all possible $X$, which often explodes for any non-superadditive risk measure.
Diversification ratio

We define a law-invariant version of diversification ratio:

$$\Delta^F_n(\rho) = \sup \left\{ \frac{\rho(X_1 + \cdots + X_n)}{\rho(X_1) + \cdots + \rho(X_n)} : X_1, \ldots, X_n \sim F \right\}.$$  

Here we assumed homogeneity in $F_i$ for:

- mathematical tractability;
- that it makes sense to let $n$ vary;
- that it characterizes the superadditivity of $\rho$ (in some sense) eventually.
Define

$$\mathcal{G}_n(F) = \{X_1 + \cdots + X_n : X_1, \ldots, X_n \sim F\}.$$ 

Let $X_F \sim F$. Then

$$\Delta_n^F(\rho) = \frac{1}{n\rho(X_F)} \sup \{\rho(S) : S \in \mathcal{G}_n(F)\}.$$ 

- Known to be a difficult problem; explicit solution for VaR (under some strong conditions) given in W., Peng and Yang (2013).

We are interested in the global superadditivity ratio

$$\Delta^F(\rho) = \sup_{n \in \mathbb{N}} \Delta^F_n(\rho) = \sup_{n \in \mathbb{N}} \frac{1}{n \rho(X_F)} \sup \{ \rho(S) : S \in \mathcal{G}_n(F) \} .$$

$$\Delta^F(\rho)$$ characterizes how superadditive $$\rho$$ be can, for a fixed $$F$$. 
We are interested in the global superadditivity ratio

\[
\Delta^F(\rho) = \sup_{n \in \mathbb{N}} \Delta^F_n(\rho) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup \left\{ \rho(S) : S \in \mathcal{S}_n(F) \right\}.
\]

\(\Delta^F(\rho)\) characterizes how superadditive \(\rho\) be can, for a fixed \(F\).

The real mathematical target:

\[
\limsup_{n \to \infty} \frac{1}{n} \sup \left\{ \rho(S) : S \in \mathcal{S}_n(F) \right\}.
\]
Definition

An extreme-aggregation measure induced by a law-invariant risk measure $\rho$ is defined as

$$\Gamma_\rho : L^0 \to [-\infty, \infty], \quad \Gamma_\rho(X_F) = \lim_{n \to \infty} \sup \frac{1}{n} \sup \{\rho(S) : S \in \mathcal{G}_n(F)\}.$$
Extreme-aggregation measure

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- $\Gamma_\rho$ quantifies the limit of $\rho$ for worst-case aggregation under dependence uncertainty.
- $\Gamma_\rho$ is a law-invariant risk measure.
Proposition

If $\rho$ is (i) positive homogeneous, (ii) comonotonic additive, or (iii) convex and zero-normal, then

$$\Gamma_\rho(X_F) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sup \{ \rho(S) : S \in \mathcal{G}_n(F) \} \geq \rho(X_F)$$

If $\rho$ is subadditive then $\Gamma_\rho \leq \rho$. If it also satisfies (i), (ii) or (iii), then $\Gamma_\rho = \rho$. 
Extreme-aggregation measure

Proposition

If $\rho$ is (i) positive homogeneous, (ii) comonotonic additive, or (iii) convex and zero-normal, then

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If $\rho$ is subadditive then $\Gamma_\rho \leq \rho$. If it also satisfies (i), (ii) or (iii), then $\Gamma_\rho = \rho$.

Remark

$\Gamma_\rho$ inherits monotonicity, cash-invariance, positive homogeneity, subadditivity, convexity, or zero-normalization from $\rho$ if $\rho$ has the corresponding properties.
Question: given an non-subadditive risk measure $\rho$, what is $\Gamma_{\rho}$?
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- Known motivating result (Wang and W., 2014): as $n \to \infty$,

$$\frac{\sup\{\text{VaR}_p(S) : S \in \mathcal{G}_n(F)\}}{\sup\{\text{ES}_p(S) : S \in \mathcal{G}_n(F)\}} \to 1.$$  

This implies $\Gamma_{\text{VaR}_p} = \Gamma_{\text{ES}_p} = \text{ES}_p$. 
Main result for distortion risk measures

Distortion risk measures:

\[ \rho(X_F) = \int_0^1 F^{-1}(t)dh(t). \]

\(h\): probability measure on \((0, 1)\). A distortion function.
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Theorem (Extreme-aggregation for distortion risk measures)

Suppose \(\rho\) is a distortion risk measure with distortion function \(h\), then \(\Gamma \rho\) is

(a) the smallest coherent risk measure dominating \(\rho\);

(b) a coherent distortion risk measure with a distortion function as the largest convex distortion function dominated by \(h\).
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- Example: \( \Gamma_{\text{VaR}_p} = \text{ES}_p \).
- A proof of more than 10 pages (single spaced).
Main result for distortion risk measures

For distortion risk measures,

\[ \Delta^F(\rho) = \frac{\Gamma_{\rho}(X_F)}{\rho(X_F)}. \]
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\[ \Delta^F(\rho) = \frac{\Gamma_\rho(X_F)}{\rho(X_F)}. \]

Theorem (Coherence and extreme-aggregation)

Suppose \( \rho \) is distortion risk measure. The following are equivalent:

(a) \( \rho \) is coherent.

(b) \( \Gamma_\rho(X_F) = \rho(X_F) \) for all distributions \( F \).

(c) \( \Gamma_\rho(X_F) = \rho(X_F) \) for some continuous distribution \( F \), \( \rho(X_F) < \infty \).

(d) \( \Delta^F(\rho) = 1 \) for all distributions \( F \), \( \rho(X_F) \in (0, \infty) \).

(e) \( \Delta^F(\rho) = 1 \) for some continuous distribution \( F \), \( \rho(X_F) \in (0, \infty) \).
Main result for shortfall risk measures

Shortfall risk measures:

\[ \rho(X) = \inf\{y \in \mathbb{R} : \mathbb{E}[\ell(X - y)] \leq l(0)\}. \]

\( \ell \): convex and increasing function. A loss function.
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**Theorem (Extreme-aggregation for shortfall risk measures)**

Suppose \( \rho \) is a shortfall risk measure with loss function \( \ell \), then \( \Gamma_\rho \) is

(a) the smallest coherent risk measure dominating \( \rho \);

(b) a coherent \( p \)-expectile, where

\[ p = \lim_{x \to \infty} \frac{\ell'(x)}{\lim_{x \to \infty} \ell'(x) + \lim_{x \to -\infty} \ell'(x)} \]
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(a) the smallest coherent risk measure dominating \( \rho \);

(b) a coherent \( p \)-expectile, where

\[ p = \lim_{x \to \infty} \ell'(x) / \left( \lim_{x \to \infty} \ell'(x) + \lim_{x \to -\infty} \ell'(x) \right) \]

- Example: \( \Gamma_{\text{ER}_\beta} = e_1 = \text{ess-sup} \), where \( \text{ER}_\beta \) is the entropy risk measure: with loss function \( \ell(x) = \exp(\beta x) - 1 \).
- A proof of one page.
Conclusion

- $\Gamma_\rho$ is always a coherent risk measure in all studied examples whenever $\rho$ is standard.
- $\Gamma_\rho$ gains positive homogeneity, convexity, and subadditivity even if $\rho$ does not have these properties, in all studied examples.
- A universal proof of this phenomenon is not available yet.
- Characterize the class of risk measures which induce coherent extreme-aggregation measures?
Conclusion

Some take-home message:

**Coherence** is indeed a natural property desired by a good risk measure. Even when a non-coherent risk measure is applied to a portfolio, its extreme behavior under dependence uncertainty leads to coherence.

This contributes to the Basel question and partly supports the use of coherent risk measures.
References I


Thank you for your kind attendance!
R1, Page 20, *Choice of risk metric*:

“... However, a number of weaknesses have been identified with VaR, including its inability to capture “tail risk”. The Committee therefore believes it is necessary to consider alternative risk metrics that may overcome these weaknesses.”
We focus on the mathematical and statistical aspects, avoiding discussion on practicalities and operational issues.

From R1, Page 3:

“The Committee recognises that moving to ES could entail certain operational challenges; nonetheless it believes that these are outweighed by the benefits of replacing VaR with a measure that better captures tail risk.”
R2, Page 3, Approach to risk management:

“the Committee has its intention to pursue two key confirmed reforms outlined in the first consultative paper [May 2012]: Stressed calibration . . . Move from Value-at-Risk (VaR) to Expected Shortfall (ES).”