Comparison of multivariate risks, new results and applications

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Outline of talk

- Stop-loss order
- Cut–criterion of the stop-loss order
- Bivariate extension : Cut–criterion of the upper orthant convex order
- Application : comparison of two different families of copulas
- Upper orthant convex order and comonotonicity

Stop-loss order

Consider two risks X and Y with given survival functions $\overline{F}$ and $\overline{G}$.

Definition : X is smaller than Y in stop-loss (or increasing convex order), denoted $X \preceq_{sl} Y$ if

$$\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$$

holds for all the increasing convex functions v such that the expectations exist.

Characterization : Denote $x_+ = \max(x, 0)$

$$X \preceq_{sl} Y \Leftrightarrow \mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+] \quad \text{for all retention} \quad d > 0.$$

Karlin–Novikoff cut–criterion

A sufficient condition of the stop-loss order is given by :

Cut-criterion (Karlin and Novikoff (1963)) : Let X and Y be two risks with $\mathbb{E}(X) \leq \mathbb{E}(Y)$. If there exists a constant c such that

$$\begin{cases} \overline{F}(x) \geq \overline{G}(x) \text{ for all } x < c, \\ \overline{F}(x) \leq \overline{G}(x) \text{ for all } x \geq c, \end{cases}$$

then

$$X \preceq_{sl} Y.$$

Bivariate orthant convex order

Definition : Given non-negative random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$. We say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the orthant convex order denoted as $\mathbf{X} \preceq_{\text{uo-cx}} \mathbf{Y}$ if the inequalities

$$\mathbb{E}[v_1(X_1)v_2(X_2)] \leq \mathbb{E}[v_1(Y_1)v_2(Y_2)]$$

holds for all non-decreasing convex functions v_1 and v_2.

Characterization : $\mathbf{X} \preceq_{\text{uo-cx}} \mathbf{Y}$ if and only, if

i) $\mathbb{E}[(X_i - d_i)_+] \leq \mathbb{E}[(Y_i - d_i)_+]$ for all $d_i > 0$, $i = 1, 2$

ii) $\mathbb{E}[(X_1 - d_1)_+(X_2 - d_2)_+] \leq \mathbb{E}[(Y_1 - d_1)_+(Y_2 - d_2)_+]$ for all $d_1, d_2 > 0$.

Link with stop-loss order

Consequently :

$$\mathbf{X} \preceq_{\text{uo-cx}} \mathbf{Y} \quad \Rightarrow \quad X_i \preceq_{sl} Y_i, \quad i = 1, 2.$$

This shows that $\preceq_{\text{uo-cx}}$ can be viewed as bivariate extension of stop-loss order.

One can wonder if there exists cut-criterion for the bivariate orthant convex order $\preceq_{\text{uo-cx}}$. The answer of this question is given in the next slide.

Crossing condition for the bivariate orthant convex order

Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be non-negative random vectors with survival functions $\bar{F}$ and $\bar{G}$. Let h be a level curve defined by

$$\bar{F}\left(x, h(x)\right) - \bar{G}\left(x, h(x)\right) = 0, \quad x \geq 0.$$

Let

$$\mathcal{C} = \left\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : y \leq h(x)\right\}.$$

We denote by $\overline{\mathcal{C}}$ the complement of $\mathcal{C}$ in $\mathbb{R}^+ \times \mathbb{R}^+$.

Extension of Karlin-Novikoff cut-criterion to the bivariate case

The conditions which leads to the bivariate orthant convex order are given below.

Bivariate cut-criterion : Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be non-negative random vectors with survival functions $\bar{F}$ and $\bar{G}$. Assume that the level curve h is decreasing. Then, if the inequalities

$$E[X_1 \mathbb{I}_{(X_2 > y)}] \leq E[Y_1 \mathbb{I}_{(Y_2 > y)}] \text{ and } E[X_2 \mathbb{I}_{(X_1 > x)}] \leq E[Y_2 \mathbb{I}_{(Y_1 > x)}]$$

hold true for all $(x, y) \in \mathcal{C}$ and

$$\begin{cases} \bar{G}(x,y) - \bar{F}(x,y) \leq 0 & \text{for all} \quad (x,y) \in \mathcal{C}, \\ \bar{G}(x,y) - \bar{F}(x,y) \geq 0 & \text{for all} \quad (x,y) \in \overline{\mathcal{C}}. \end{cases}$$

Then,

$$\mathbf{X} \preceq_{\text{uo-cx}} \mathbf{Y}.$$

Extension of Karlin-Novikoff cut-criterion to the bivariate case

Remark : For random vectors $\mathbf{X}$ and $\mathbf{Y}$ with same marginal distributions : $X_1 =_d Y_1$ and $X_2 =_d Y_2$, the inequalities

$$E[X_1 \mathbb{I}_{(X_2 > y)}] \leq E[Y_1 \mathbb{I}_{(Y_2 > y)}] \text{ and } E[X_2 \mathbb{I}_{(X_1 > x)}] \leq E[Y_2 \mathbb{I}_{(Y_1 > x)}]$$

are equivalent to conditional expectations inequalities :

$$E[X_1 | X_2 > y] \leq E[Y_1 | Y_2 > y] \text{ and } E[X_2 | X_1 > y] \leq E[Y_2 | Y_1 > y]$$

Link with concordance order

In the set of all copulas, the concordance order and the upper orthant convex are defined by

$$C_1 \preceq_C C_2 \quad \Leftrightarrow \quad C_1(u,v) \leq C_2(u,v) \quad \text{for every} \quad u,v \in [0,1]$$

and

$$C_1 \preceq_{\text{uo-cx}} C_2 \quad \Leftrightarrow \quad \int_u^1 \int_v^1 C_1(x,y)dxdy \leq \int_u^1 \int_v^1 C_1(x,y)dxdy \quad u,v \in [0,1].$$

From these definition, we see that the upper orthant convex is weaker than the concordance order, that is,

$$C_{\theta_1} \preceq_C C_{\theta_2} \quad \Rightarrow \quad C_{\theta_1} \preceq_{\text{uo-cx}} C_{\theta_2}.$$

Application : comparison of two families of copulas

The concordance order is used to compare members of a given copula family C_θ when the dependence parameter varies :

$$\theta_1 \leq \theta_2 \quad \Rightarrow \quad C_{\theta_1} \preceq_C C_{\theta_2}.$$

In general, there is no comparison between a copulas from different families with $\preceq_C$:

$$C_{\theta_1} \not\preceq_C C_{\theta_2} \quad \text{and} \quad C_{\theta_2} \not\preceq_C C_{\theta_1}.$$

Application : comparison of two families of copulas

For example, let C_{θ_1} be a Clayton copula with parameter $\theta_1 = 1$ and C_{θ_2} be a Frank copula with parameter $\theta_2 = 2$. The next graphic shows that the quantity $C_{\theta_2}(u, v) - C_{\theta_1}(u, v)$ changes sign over the unit square.

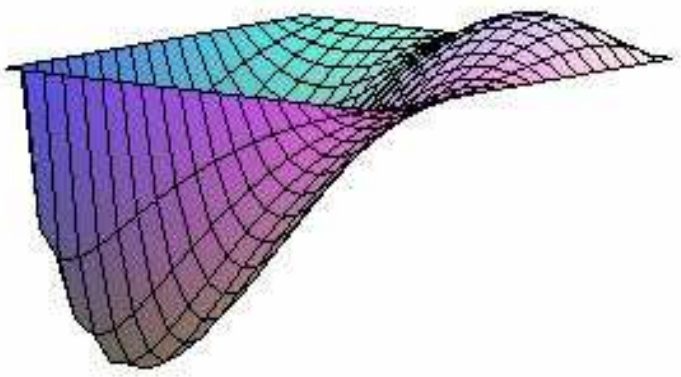

FIGURE : Graph of $(u, v) \mapsto C_{\theta_2}(u, v) - C_{\theta_1}(u, v)$

Application : comparison of two families of copulas

Since $\preceq_{\text{uo-cx}}$ is weaker than $\preceq_C$. Thus one can expect to rank the copulas C_{θ_1} and C_{θ_2} with respect to $\preceq_{\text{uo-cx}}$ instead of $\preceq_C$.

To this end, one can use our cut-criterion to establish a such comparison with respect $\preceq_{\text{uo-cx}}$. For that, it suffices to verify the next conditions.

Application : comparison of two families of copulas

- Step 1 : the level curve $v = h(u)$ defined by

$$C_2(u, h(u)) - C_1(u, h(u)) = 0$$

should be decreasing.

- Step 2 : one has to verify that

$$\begin{cases} C_{\theta_2}(u, v) - C_{\theta_1}(u, v) \leq 0 & \text{if} \quad v \leq h(u), \\ C_{\theta_2}(u, v) - C_{\theta_1}(u, v) \geq 0 & \text{if} \quad v \geq h(u). \end{cases}$$

- Step 3 : Verify the moment conditions :
$g(v) = E[U_1 | V_1 > v] - E[U_2 | V_2 > v] \leq 0$ for all $v \in [0, 1]$.

Application : comparison of two families of copulas

Steps 1–2 : the level curve is decreasing as described in the next figure.

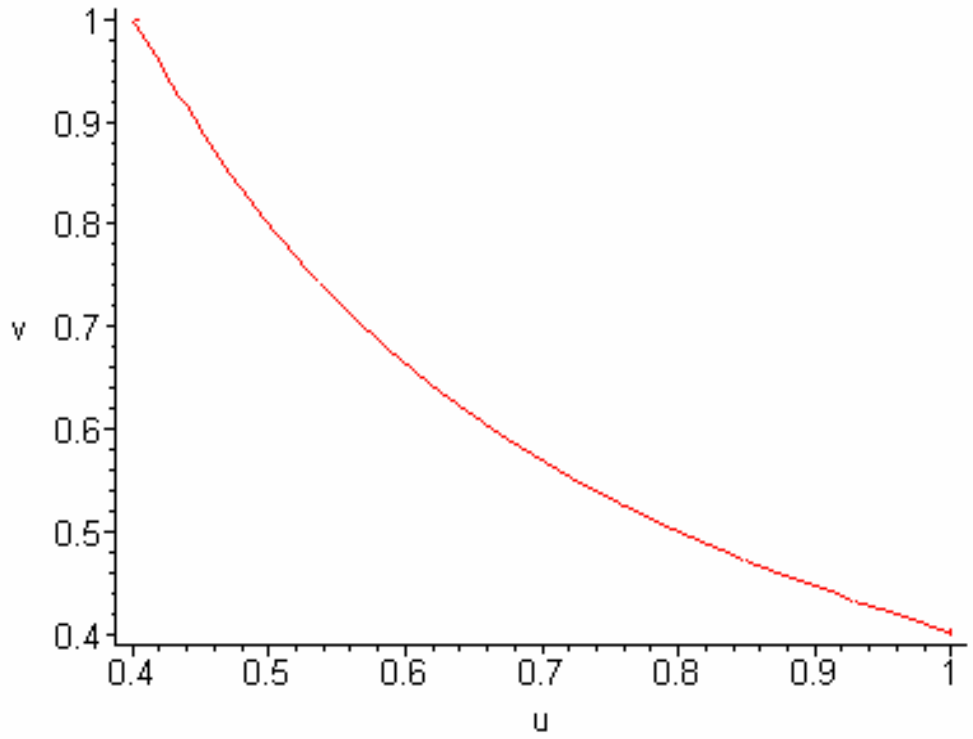

FIGURE : Graph of the level curve $v = h(u)$

Application : comparison of two families of copulas

Step 3 : the function $v \mapsto g(v) = E[U_1|V_1 > v] - E[U_2|V_2 > v]$ displayed in next figure is indeed negative over $[0, 1]$. We then conclude that $C_{\theta_1} \preceq_{\text{uo-cx}} C_{\theta_2}$.

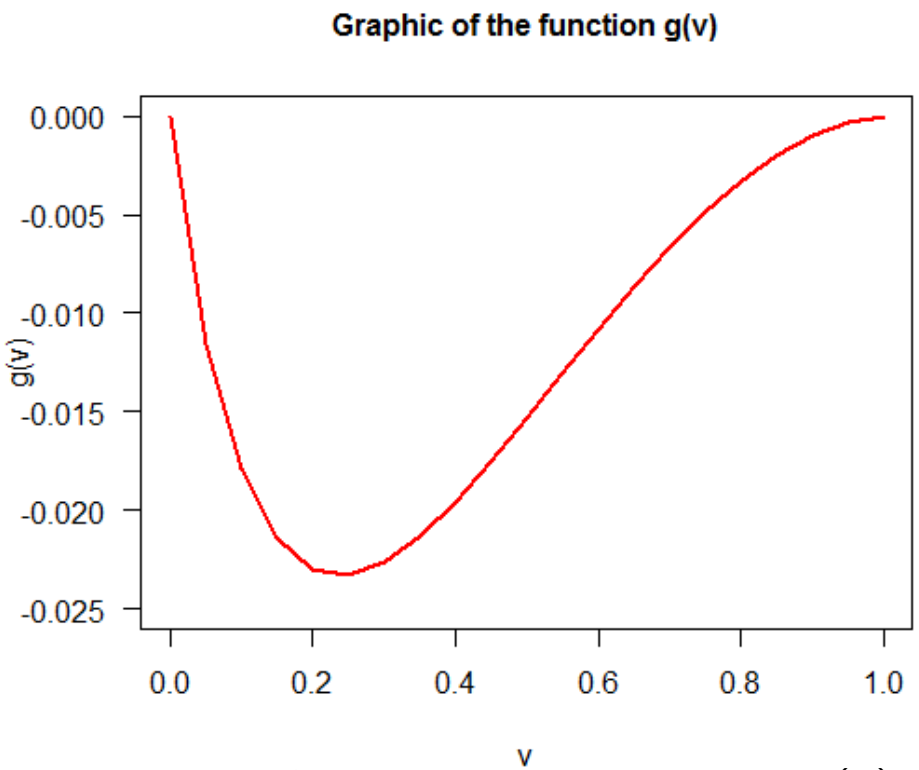

FIGURE : Graph of the function $v = g(u)$

Application : comparison of two families of copulas

- **Conclusion :** We see that $C_{\theta_1} \preceq_{\text{uo-cx}} C_{\theta_2}$. This means that the upper orthant convex order can be more convenient for compare the concordance between two different families of copulas.

Stop-loss order and comonotonicity

- Comonotonicity

  The random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is said to be comonotonic, if its components are perfectly dependent : $Y_i = G_i^{-1}(V)$, where G_i is the distribution function of Y_i and $V \sim \mathcal{U}_{[0,1]}$.

  In other words, the copula of $\mathbf{Y}$ is exactly the Fréchet upper bound :

$$M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n), \quad u_i \in [0, 1].$$

Stop-loss order and comonotonicity

Following Dhaene et al. (2002) :

If $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are two sets of random variables such that :

- $X_i \preceq_{sl} Y_i$ for all i
- $Y_1, \ldots, Y_n$ are comonotonic

then

$$\sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} Y_i.$$

In particular, if $X_i \sim F_i$ and $U \sim [0,1]$, then

$$\sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} F_i^{-1}(U).$$

Notice that this inequality provides many applications in actuarial science.

Upper orthant order and comonotonicity

Let us now switch to the multivariate case and consider random variables

- d-class of random variables $X_{1j}, X_{2j}, \ldots, X_{nj}$, $j = 1, \ldots, d$
- d-class of random variables $Y_{1j}, X_{2j}, \ldots, X_{nj}$, $j = 1, \ldots, d$
- Assume that $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ for $j = 1, \ldots, d$ form d sets of comonotonic random variables

What can we say about the ordering of the random vectors

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \text{ and } \left(\sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right).$$

Upper orthant order and comonotonicity

- First, remark the following decomposition :

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id}\right) = \frac{1}{n^{d-1}} \sum_{\mathbf{i} \in \{1,\ldots,n\}^d} \mathbf{X}_{\mathbf{i}}.$$

where :

$$\mathbf{X}_{\mathbf{i}} = (X_{i_1 1}, \ldots, X_{i_d d}) \quad \mathbf{i} = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d.$$

- This decomposition show that $\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id}\right)$ is affected by the random vectors $\mathbf{X}_{\mathbf{i}}$, $\mathbf{i} = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d$.

- Example :

$$(X_{11}+X_{21}, X_{12}+X_{22}) = \frac{1}{2}\left[(X_{11}, X_{12}) + (X_{11}, X_{22}) + (X_{21}, X_{12}) + (X_{21}, X_{22})\right].$$

Upper orthant order and comonotonicity

Proposition : Let X_{ij} and Y_{ij}, $j = 1, \ldots, d$, $i = 1, \ldots, n$, be two arrays of random variables such that

- $Y_{1j}, \ldots, Y_{nj}$ are comonotonic for every $j = 1, \ldots, d$
- $\mathbf{X_i} \preceq_{uo-cx} \mathbf{Y_i}$ for all $\mathbf{i} \in \{1, \ldots, n\}^d$.

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{uo-cx} \left(\sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right).$$

- For $d = 1$, one obtain the result of Dhaene et al. (2002) :

$$X_i \preceq_{sl} Y_i \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} Y_i.$$

Conditionally increasing

Let us recall some strong positive dependence structure : a random vector $(X_1, \ldots, X_d)$ is said to be conditionally increasing (CI) if

$$E[g(X_i)|X_j = x_j, \quad j \in J]$$

is non-decreasing in x_j, $j \in J$ for every $J \subset \{1, \ldots, d\}$, $i \notin J$, and non-decreasing function g for which the expectation is defined, $i = 2, 3, \ldots, d$.

If $\mathbf{X_i}$ and $\mathbf{Y_i}$ are conditionally increasing, then we can replace

- $\mathbf{X_i} \preceq_{uo-cx} \mathbf{Y_i}$ for all $\mathbf{i} \in \{1, \ldots, n\}^d$.

by easier stochastic inequalities, namely

- $X_{ij} \preceq_{sl} Y_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, d$.

Upper orthant order and comonotonicity

Proposition : Let X_{ij} be continuous random variables with distribution functions F_{ij}, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_{\mathbf{i}}$ be the copula associated to the random vectors $\mathbf{X_i}$, $\mathbf{i} \in \{1, \ldots, n\}^d$. Assume that there exists a copula C_U such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_{\mathbf{i}}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d).$$

Let $(U_1, \ldots, U_d)$ be a random vector distributed as C_U. Then

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{\text{uo-cx}} \left(\sum_{i=1}^{n} F_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^{n} F_{id}^{-1}(U_d) \right).$$

- For $d = 1$, one obtain the result of Dhaene et al. (2002) :

$$\sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} F_i^{-1}(U).$$

Upper orthant order and comonotonicity

Proposition : Let X_{ij} and Y_{ij} be continuous random variables with distribution functions F_{ij} and G_{ij}, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_{\mathbf{i}}$ be the copula associated to the random vectors $\mathbf{X_i}$, $\mathbf{i} \in \{1, \ldots, n\}^d$. Assume that there exists a copula C_U such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_{\mathbf{i}}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$$

Assume that the random vector $(U_1, \ldots, U_d)$ distributed as C_U is CI. Then

$$X_{ij} \preceq_{sl} Y_{ij}$$
$$\Downarrow$$

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id}\right) \preceq_{\text{uo-cx}} \left(\sum_{i=1}^{n} G_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^{n} G_{id}^{-1}(U_d)\right).$$

Comparison of multi-class portfolio

Let us recall the notion of the s-increasing convex orderings, denoted $\preceq_{s-icx}$.

Let X and Y be two risks : $X \preceq_{s-icx} Y$ if

$$E(\phi(X)) \leq E(\phi(Y))$$

for all ϕ such that

$$\frac{d^k \phi}{dx^k}(x) \geq 0.$$

Lemma : For any random vectors $\mathbf{X} = (X_1, \ldots, X_d)$ and $\mathbf{Y} = (Y_1, \ldots, Y_d)$

$$\mathbf{X} \preceq_{\text{uo-cx}} \mathbf{Y} \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \preceq_{2d-icx} \sum_{i=1}^{n} Y_i.$$

Comparison of multi-class portfolio

Corollary : Let X_{ij} and Y_{ij} be continuous random variables with distribution functions F_{ij} and G_{ij}, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_{\mathbf{i}}$ be the copula associated to the random vectors $\mathbf{X_i}$, $\mathbf{i} \in \{1, \ldots, n\}^d$. Assume that there exists a copula C_U such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_{\mathbf{i}}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$$

Assume that the random vector $(U_1, \ldots, U_d)$ distributed as C_U is CI. Then

$$X_{ij} \preceq_{sl} Y_{ij}$$
$$\Downarrow$$
$$\sum_{i=1}^{n} \sum_{j=1}^{d} X_{ij} \preceq_{2d-icx} \sum_{i=1}^{n} \sum_{j=1}^{d} G_{ij}^{-1}(U_j).$$