Comparison of multivariate risks, new results and applications

Mesfioui Mhamed

Université du Quebec de Trois-Rivières

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Outline of talk

- Stop-loss order
- Cut–criterion of the stop-loss order
- Bivariate extension: Cut–criterion of the upper orthant convex order
- Application: comparison of two different families of copulas
- Upper orthant convex order and comonotonicity
Stop-loss order

Consider two risks $X$ and $Y$ with given survival functions $\overline{F}$ and $\overline{G}$.

**Definition**: $X$ is smaller than $Y$ in stop-loss (or increasing convex order), denoted $X \preceq_{sl} Y$ if

$$E[v(X)] \leq E[v(Y)]$$

holds for all the increasing convex functions $v$ such that the expectations exist.

**Characterization**: Denote $x_+ = \max(x, 0)$

$$X \preceq_{sl} Y \iff E[(X - d)_+] \leq E[(Y - d)_+] \quad \text{for all retention } d > 0.$$
A sufficient condition of the stop-loss order is given by:

**Cut-criterion (Karlin and Novikoff (1963))**: Let $X$ and $Y$ be two risks with $E(X) \leq E(Y)$. If there exists a constant $c$ such that

\[
\begin{align*}
F(x) &\geq G(x) \text{ for all } x < c, \\
F(x) &\leq G(x) \text{ for all } x \geq c,
\end{align*}
\]

then

\[ X \preceq_{sl} Y. \]
Bivariate orthant convex order

**Definition**: Given non-negative random vectors \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2) \). We say that \( X \) is smaller than \( Y \) in the orthant convex order denoted as \( X \preceq_{\text{uo-cx}} Y \) if the inequalities

\[
E[v_1(X_1)v_2(X_2)] \leq E[v_1(Y_1)v_2(Y_2)]
\]

holds for all non-decreasing convex functions \( v_1 \) and \( v_2 \).

**Characterization**: \( X \preceq_{\text{uo-cx}} Y \) if and only if

i) \( E[(X_i - d_i)_+] \leq E[(Y_i - d_i)_+] \) for all \( d_i > 0, i = 1, 2 \)

ii) \( E[(X_1 - d_1)_+(X_2 - d_2)_+] \leq E[(Y_1 - d_1)_+(Y_2 - d_2)_+] \) for all \( d_1, d_2 > 0 \).
Consequently:

\[ X \preceq_{uo-cx} Y \quad \Rightarrow \quad X_i \preceq_{sl} Y_i, \quad i = 1, 2. \]

This shows that \( \preceq_{uo-cx} \) can be viewed as bivariate extension of stop-loss order.

One can wonder if there exists cut-criterion for the bivariate orthant convex order \( \preceq_{uo-cx} \). The answer of this question is given in the next slide.
Crossing condition for the bivariate orthant convex order

Let \( \mathbf{X} = (X_1, X_2) \) and \( \mathbf{Y} = (Y_1, Y_2) \) be non-negative random vectors with survival functions \( \bar{F} \) and \( \bar{G} \). Let \( h \) be a level curve defined by

\[
\bar{F}(x, h(x)) - \bar{G}(x, h(x)) = 0, \quad x \geq 0.
\]

Let

\[
\mathcal{C} = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : y \leq h(x)\}.
\]

We denote by \( \overline{\mathcal{C}} \) the complement of \( \mathcal{C} \) in \( \mathbb{R}^+ \times \mathbb{R}^+ \).
Extension of Karlin-Novikoff cut-criterion to the bivariate case

The conditions which leads to the bivariate orthant convex order are given below.

**Bivariate cut-criterion**: Let \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2) \) be non-negative random vectors with survival functions \( \bar{F} \) and \( \bar{G} \). Assume that the level curve \( h \) is decreasing. Then, if the inequalities

\[
E[X_1 \mathbb{1}(X_2 > y)] \leq E[Y_1 \mathbb{1}(Y_2 > y)] \quad \text{and} \quad E[X_2 \mathbb{1}(X_1 > x)] \leq E[Y_2 \mathbb{1}(Y_1 > x)]
\]

hold true for all \((x, y) \in \mathcal{C}\) and

\[
\left\{ \begin{array}{l}
\bar{G}(x, y) - \bar{F}(x, y) \leq 0 \quad \text{for all} \quad (x, y) \in \mathcal{C}, \\
\bar{G}(x, y) - \bar{F}(x, y) \geq 0 \quad \text{for all} \quad (x, y) \in \bar{\mathcal{C}}.
\end{array} \right.
\]

Then,

\[ X \preceq_{\text{uo-cx}} Y. \]
Remark: For random vectors $X$ and $Y$ with same marginal distributions: $X_1 =_d Y_1$ and $X_2 =_d Y_2$, the inequalities

$$E[X_1 \mathbb{I}(X_2 > y)] \leq E[Y_1 \mathbb{I}(Y_2 > y)] \text{ and } E[X_2 \mathbb{I}(X_1 > x)] \leq E[Y_2 \mathbb{I}(Y_1 > x)]$$

are equivalent to conditional expectations inequalities:

$$E[X_1 | X_2 > y] \leq E[Y_1 | Y_2 > y] \text{ and } E[X_2 | X_1 > y] \leq E[Y_2 | Y_1 > y]$$
Link with concordance order

In the set of all copulas, the concordance order and the upper orthant convex are defined by

\[ C_1 \preceq C_2 \iff C_1(u, v) \leq C_2(u, v) \quad \text{for every} \quad u, v \in [0, 1] \]

and

\[ C_1 \preceq_{uo-\text{cx}} C_2 \iff \int_0^1 \int_0^1 C_1(x, y) \, dx \, dy \leq \int_0^1 \int_0^1 C_1(x, y) \, dx \, dy \quad u, v \in [0, 1]. \]

From these definition, we see that the upper orthant convex is weaker than the concordance order, that is,

\[ C_{\theta_1} \preceq C_{\theta_2} \implies C_{\theta_1} \preceq_{uo-\text{cx}} C_{\theta_2}. \]
The concordance order is used to compare members of a given copula family $C_\theta$ when the dependence parameter varies:

$$\theta_1 \leq \theta_2 \Rightarrow C_{\theta_1} \preceq C_{\theta_2}.$$ 

In general, there is no comparison between a copulas from different families with $\preceq_C$:

$$C_{\theta_1} \npreceq C_{\theta_2} \quad \text{and} \quad C_{\theta_2} \npreceq C_{\theta_1}.$$
Application: comparison of two families of copulas

For example, let $C_{\theta_1}$ be a Clayton copula with parameter $\theta_1 = 1$ and $C_{\theta_2}$ be a Frank copula with parameter $\theta_2 = 2$. The next graphic shows that the quantity $C_{\theta_2}(u, v) - C_{\theta_1}(u, v)$ changes sign over the unit square.

**Figure:** Graph of $(u, v) \mapsto C_{\theta_2}(u, v) - C_{\theta_1}(u, v)$
Since $\preceq_{uo-cx}$ is weaker than $\preceq_C$. Thus one can expect to rank the copulas $C_{\theta_1}$ and $C_{\theta_2}$ with respect to $\preceq_{uo-cx}$ instead of $\preceq_C$.

To this end, one can use our cut-criterion to establish a such comparison with respect $\preceq_{uo-cx}$. For that, it suffices to verify the next conditions.
Application: comparison of two families of copulas

- **Step 1:** the level curve $v = h(u)$ defined by
  \[ C_2(u, h(u)) - C_1(u, h(u)) = 0 \]
  should be decreasing.

- **Step 2:** one has to verify that
  \[
  \begin{cases}
  C_{\theta_2}(u, v) - C_{\theta_1}(u, v) \leq 0 & \text{if } v \leq h(u), \\
  C_{\theta_2}(u, v) - C_{\theta_1}(u, v) \geq 0 & \text{if } v \geq h(u).
  \end{cases}
  \]

- **Step 3:** Verify the moment conditions:
  \[ g(v) = E[U_1 | V_1 > v] - E[U_2 | V_2 > v] \leq 0 \text{ for all } v \in [0, 1]. \]
Application: comparison of two families of copulas

Steps 1–2: the level curve is decreasing as described in the next figure.

**Figure:** Graph of the level curve $v = h(u)$
Application: comparison of two families of copulas

Step 3: the function \( v \mapsto g(v) = E[U_1|V_1 > v] - E[U_2|V_2 > v] \) displayed in next figure is indeed negative over \([0, 1]\). We then conclude that \( C_{\theta_1} \preceq_{uo-cx} C_{\theta_2} \).

**Figure**: Graph of the function \( v = g(u) \)
Conclusion: We see that $C_{\theta_1} \leq_{uo-cx} C_{\theta_2}$. This means that the upper orthant convex order can be more convenient for compare the concordance between two different families of copulas.
Stop-loss order and comonotonicity

- Comonotonicity

The random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is said to be comonotonic, if its components are perfectly dependent: $Y_i = G_i^{-1}(V)$, where $G_i$ is the distribution function of $Y_i$ and $V \sim \mathcal{U}_{[0,1]}$.

In other words, the copula of $\mathbf{Y}$ is exactly the Fréchet upper bound:

$$M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n), \quad u_i \in [0,1].$$
Stop-loss order and comonotonicity

Following Dhaene et al. (2002):

If $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are two sets of random variables such that:

- $X_i \preceq_{sl} Y_i$ for all $i$
- $Y_1, \ldots, Y_n$ are comonotonic

then

$$\sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} Y_i.$$ 

In particular, if $X_i \sim F_i$ and $U \sim [0, 1]$, then

$$\sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} F_i^{-1}(U).$$

Notice that this inequality provides many applications in actuarial science.
Upper orthant order and comonotonicity

Let us now switch to the multivariate case and consider random variables

- $d$-class of random variables $X_{1j}, X_{2j}, \ldots, X_{nj}, j = 1, \ldots, d$
- $d$-class of random variables $Y_{1j}, X_{2j}, \ldots, X_{nj}, j = 1, \ldots, d$
- Assume that $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ for $j = 1, \ldots, d$ form $d$ sets of comonotonic random variables

What can we say about the ordering of the random vectors

$$\left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \text{ and } \left( \sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right).$$
Upper orthant order and comonotonicity

First, remark the following decomposition:

\[
\left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) = \frac{1}{n^{d-1}} \sum_{i \in \{1, \ldots, n\}^d} X_i.
\]

where:

\[X_i = (X_{i1}, \ldots, X_{id}) \quad i = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d.\]

This decomposition show that \( (\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id}) \) is affected by the random vectors \( X_i, i = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d.\)

Example:

\[
(X_{11} + X_{21}, X_{12} + X_{22}) = \frac{1}{2} \left[ (X_{11}, X_{12}) + (X_{11}, X_{22}) + (X_{21}, X_{12}) + (X_{21}, X_{22}) \right].
\]
**Proposition** : Let $X_{ij}$ and $Y_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$, be two arrays of random variables such that

- $Y_{1j}, \ldots, Y_{nj}$ are comonotonic for every $j = 1, \ldots, d$
- $X_i \preceq_{uo-cx} Y_i$ for all $i \in \{1, \ldots, n\}^d$.

\[
\left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{uo-cx} \left( \sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right).
\]

For $d = 1$, one obtain the result of Dhaene et al. (2002) :

\[
X_i \preceq_{sl} Y_i \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \preceq_{sl} \sum_{i=1}^{n} Y_i.
\]
Let us recall some strong positive dependence structure: a random vector $(X_1, \ldots, X_d)$ is said to be conditionally increasing (CI) if

$$E[g(X_i)|X_j = x_j, \ j \in J]$$

is non-decreasing in $x_j, j \in J$ for every $J \subset \{1, \ldots, d\}, i \not\in J$, and non-decreasing function $g$ for which the expectation is defined, $i = 2, 3, \ldots, d$.

If $X_i$ and $Y_i$ are conditionally increasing, then we can replace

- $X_i \preceq_{uo-cx} Y_i$ for all $i \in \{1, \ldots, n\}^d$.

by easier stochastic inequalities, namely

- $X_{ij} \preceq_{sl} Y_{ij}, i = 1, \ldots, n, j = 1, \ldots, d$. 
Upper orthant order and comonotonicity

**Proposition**: Let $X_{ij}$ be continuous random variables with distribution functions $F_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_i$ be the copula associated to the random vectors $X_i$, $i \in \{1, \ldots, n\}^d$. Assume that there exists a copula $C_U$ such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_i(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d).$$

Let $(U_1, \ldots, U_d)$ be a random vector distributed as $C_U$. Then

$$\left( \sum_{i=1}^n X_{i1}, \ldots, \sum_{i=1}^n X_{id} \right) \lesssim_{uo-cx} \left( \sum_{i=1}^n F_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^n F_{id}^{-1}(U_d) \right).$$

For $d = 1$, one obtain the result of Dhaene et al. (2002):

$$\sum_{i=1}^n X_i \lesssim_{sl} \sum_{i=1}^n F_{i1}^{-1}(U).$$
Upper orthant order and comonotonicity

**Proposition**: Let $X_{ij}$ and $Y_{ij}$ be continuous random variables with distribution functions $F_{ij}$ and $G_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_i$ be the copula associated to the random vectors $X_i$, $i \in \{1, \ldots, n\}^d$. Assume that there exists a copula $C_U$ such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_i(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$$

Assume that the random vector $(U_1, \ldots, U_d)$ distributed as $C_U$ is CI. Then

$$X_{ij} \preceq_{sl} Y_{ij}$$

$$\downarrow$$

$$\left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{uo-cx} \left( \sum_{i=1}^{n} G_i^{-1}(U_1), \ldots, \sum_{i=1}^{n} G_i^{-1}(U_d) \right).$$
Comparison of multi-class portfolio

Let us recall the notion of the $s$-increasing convex orderings, denoted $\preceq_{s-icx}$.

Let $X$ and $Y$ be two risks: $X \preceq_{s-icx} Y$ if

$$E(\phi(X)) \leq E(\phi(Y))$$

for all $\phi$ such that

$$\frac{d^k \phi}{dx^k}(x) \geq 0.$$

Lemma: For any random vectors $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$

$$X \preceq_{uo-cx} Y \Rightarrow \sum_{i=1}^{n} X_i \preceq_{2d-icx} \sum_{i=1}^{n} Y_i.$$
Corollary: Let $X_{ij}$ and $Y_{ij}$ be continuous random variables with distribution functions $F_{ij}$ and $G_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_i$ be the copula associated to the random vectors $X_i$, $i \in \{1, \ldots, n\}^d$. Assume that there exists a copula $C_U$ such that for all $u_1, \ldots, u_d \in [0, 1]$

$$C_i(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$$

Assume that the random vector $(U_1, \ldots, U_d)$ distributed as $C_U$ is CI. Then

$$X_{ij} \preceq_{sl} Y_{ij}$$

$$\downarrow$$

$$\sum_{i=1}^{n} \sum_{j=1}^{d} X_{ij} \preceq_{2d-icx} \sum_{i=1}^{n} \sum_{j=1}^{d} G_{ij}^{-1}(U_j).$$